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COMMENT

Comments on the transfer-matrix approach to the one-dimensional bond percolation with further neighbour bonds

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Abstract. The transfer matrices which have been used in our previous studies on the one-dimensional bond percolation problem with bonds connecting Lth nearest neighbours are only for the case of directed bonds. For the case of undirected bonds, the transfer matrices are non-systematic and become much more complicated when $L \geqslant 3$. Here we study the cases of $L = 2$, 3 and find that the critical behaviour remains unchanged. We strongly suggest that this is true for any L. Rigorous evidence is given to support this conjecture.

Recently, the transfer-matrix method has been used to study the critical behaviour of the one-dimensional bond percolation problem with bonds connecting Lth nearest neighbours (Zhang and Shen 1982, Zhang *et al* 1983a, b). Very rich critical behaviour has been found in such systems. When L is finite, exact critical surface and correlation length exponents ν are found. The values of ν vary from 1, 2, ... to $L(L+1)/2$ depending on where the critical surface is approached. When L is infinite, the problem of long-range percolation has also been discussed for the distribution $p_n = p_1/n^s$ where p_n is the occupation probability of the nth nearest neighbour bond. Actually, in all these works the transfer matrices used are only for systems with directed bonds. When the bonds are not directed the transfer matrices are non-systematic and become much more complicated for $L \geqslant 3$. Here we consider only the cases of $L = 2$ and 3.

We follow the same definitions and notations of the transfer matrix used in Zhang and Shen (1982, hereafter referred to as ZS). When $L = 2$, a column consists of two sites $(i, i+1)$. We use '$\bigcirc$' or '$\times$' to denote whether a site is connected to or disconnected from the origin. The transfer matrix $M_{mn}^{(b,2)}$ which transfers the probability of a configuration n in the $(i, i+1)$ column to another configuration m in the $(i+1, i+2)$ column is given by

$$M_{mn}^{(b,2)} = \begin{pmatrix} & (\times,\times) & (\bigcirc,\times) & (\times,\bigcirc) & (\bigcirc,\bigcirc) \\ & 1 & q_2 & 0 & 0 \\ & 0 & 0 & q_1 & q_1q_2 \\ & 0 & q_1p_2 & 0 & 0 \\ & 0 & p_1p_2 & p_1 & 1-q_1q_2 \end{pmatrix} \qquad (1)$$

where $q_i = 1 - p_i$, $i = 1, 2$. Notice that the elements $M_{32}^{(b,2)}$ and $M_{42}^{(b,2)}$ in (1) are different from those of (8) in ZS. Not only this, when $L \geqslant 3$, we have to consider those configurations where the disconnected sites might connect among themselves (Essam

$$
M_{mn}^{(b,3)} = \begin{array}{ccccccccccc}
(\times,\times,\times) & (\bigcirc,\times,\times) & (\bigcirc,\widehat{\times,\times}) & (\times,\bigcirc,\times) & (\widehat{\times,\bigcirc,\times}) & (\times,\times,\bigcirc) & (\widehat{\times,\times},\bigcirc) & (\bigcirc,\bigcirc,\times) & (\bigcirc,\times,\bigcirc) & (\times,\bigcirc,\bigcirc) & (\bigcirc,\bigcirc,\bigcirc) \\
\end{array}
$$

$$
\begin{pmatrix}
1 & q_3 & q_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & q_1q_2 & q_1q_2 & 0 & 0 & q_1q_2q_3 & 0 & 0 & 0 \\
0 & 0 & 0 & p_1q_2 & p_1q_2 & 0 & 0 & p_1q_2q_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & q_1q_2 & q_1q_2 & 0 & q_1q_2q_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & q_1p_2 & q_1p_2 & 0 & q_1p_2q_3 & 0 & 0 \\
0 & q_1q_2p_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & q_1q_2p_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_1q_2 & q_1q_2q_3 \\
0 & q_1p_2p_3 & 0 & q_1p_2 & q_1p_2 & 0 & 0 & q_1(1-q_2q_3) & 0 & 0 & 0 \\
0 & p_1q_2p_3 & 0 & 0 & 0 & p_1q_2 & p_2q_2 & 0 & (1-q_1q_3)q_2 & 0 & 0 \\
0 & p_1p_2p_3 & (1-q_1q_2)p_3 & p_1p_2 & p_1p_2 & p_1p_2 & p_1p_2 & p_1(1-q_2q_3) & (1-q_1q_3)p_2 & 1-q_1q_2 & 1-q_1q_2q_3
\end{pmatrix} \qquad (2)
$$

1980, Derrida and Vannimenus 1980). For $L=3$ those configurations are denoted by $(\bigcirc, \widehat{\times, \times})$, $(\widehat{\times, \bigcirc, \times})$ and $(\widehat{\times, \times}, \bigcirc)$. Equation (10) of ZS is now replaced by an 11×11 matrix.

To study the critical behaviour from the transfer matrices (1) and (2), we only have to consider the case when the chain is not decoupled (Zhang *et al* 1983b, hereafter referred to as ZPL). In general, we define a function $A^{(L)}(\lambda; \boldsymbol{q})$ by

$$\det(M_{mn}^{(b,L)}-\lambda I)\equiv(1-\lambda)A^{(L)}(\lambda;\boldsymbol{q}) \tag{3}$$

where $\boldsymbol{q}=(q_1, q_2, \ldots, q_L)$. Using the Perron–Frobenius theorem (Rosenblatt 1962), the maximum non-trivial eigenvalue $\lambda_m^{(L)}$ of $M_{mn}^{(b,L)}$ is non-degenerate and given by (ZPL)

$$\lambda_m^{(L)}=1+[A^L(1;\boldsymbol{q})/B^{(L)}(1;\boldsymbol{q})] \tag{4}$$

with

$$B^{(L)}(1;\boldsymbol{q})=(\lambda_1^{(L)}-1)(\lambda_2^{(L)}-1)\ldots(\lambda_{s(L)}^{(L)}-1) \tag{5}$$

where $\lambda_i^{(L)}$ $(i=1,2,\ldots,s(L))$ are the other eigenvalues of $M_{mn}^{(b,L)}$ besides 1 and $\lambda_m^{(L)}$. Since $M_{mn}^{(b,L)}$ is a stochastic matrix, we have $|\lambda_i^{(L)}|\leq 1$ (Pearl 1973). Thus $B^{(L)}(1,\boldsymbol{q})$ cannot be zero $(\lambda_i^{(L)}\neq 1)$ and is positive or negative depending on whether the order of $M_{mn}^{(L,b)}$ is even or odd. The correlation length $\xi^{(L)}$ is related to $\lambda_m^{(L)}$ by (Derrida and Vannimenus 1980)

$$\xi^{(L)}=-1/\ln\lambda_m^{(L)}. \tag{6}$$

From (1)–(6), the following results are easily obtained:

$$\xi^{(2)}\approx(q_1q_2^2)^{-1}B^{(2)}(1;q_1,q_2), \qquad B^{(2)}(1;q_1,q_2)>0, \tag{7}$$

and

$$\xi^{(3)}\approx-(q_1q_2^2q_3^3)^{-1}B^{(3)}(1;q_1,q_2,q_3)/[1+q_1^2(1-q_3)], \qquad B^{(3)}(1;q_1,q_2,q_3)<0. \tag{8}$$

Comparing (7) and (8) with the results of ZPL (equation (8)), we find that the critical behaviour is the same no matter whether the bonds are directed or not. This is because the system is one dimensional.

The one-dimensional system (any finite L) goes critical only when at least one of the p_i $(i=1,2,\ldots,L)$ approaches one. In this limit, whether the bonds are directed or not becomes irrelevant to the critical exponents. Thus, we strongly suggest that this is true for any finite L. One rigorous piece of evidence supporting this conjecture is the case when all the p_i's are equal. In this case, infinite cell-to-cell renormalisation group calculations for the undirected bonds (Li *et al* 1983) give $\nu=L(L+1)/2$ which agrees with the results of the transfer-matrix method for the directed bonds ((8) of ZPL).

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